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Exact Solution of 1D Asymmetric Exclusion Model with Variable Cluster Size

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An one dimensional model for an open system with two kinds of particles which are driven in opposite directions by an external field is suggested. An exact solution for a steady state is given for the low density regime and it is shown that the model possesses what might be considered a phase transition from a gaseous to a liquid state. The relation to models with a fixed number of particles on a ring is discussed.

KEY WORDS: Driven lattice gas; phase transition; condensation.

1. INTRODUCTION

The idea is to consider clusters with a varying number of particles. We have a one dimensional system with two kinds of particles which enter the cluster from opposite directions and then leave the cluster at the other end. If the particles arrive too fast to allow them to pass through the cluster as fast they enter, then the size of the cluster will grow steadily. Otherwise, one has an equilibrium size distribution. One of the main results is the determination of the condition on the entering rates which separates the two possibilities. The nature of the shift resembles in many respects the gas-liquid phase transition. The article depends heavily on the article by Derrida *et al.*[3]

2. THE MODEL

The dynamics in the interior of the cluster is the same as in Derrida *et al.*[3] However, in order to emphasize the cluster we shall talk about two kinds of particles, A and B, rather than particles and holes. We shall assume that the A-particles move only from left towards right, while the

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B-particles move only from right towards left. When an A-particle has a B-particle as its neighbour to the right, then two particles can interchange position. We shall normalize the unit of time so that the average frequency is 1 per pair and unit of time, i.e., in a small time interval of length dt the probability for an interchange is dt if the A-particle is to the left of the B-particle. If the B-particle is to left of the A-particle the probability for an interchange is zero.

The difference comes at the ends where the particles arrive and leave independently. In a time interval of length dt an A-particle is added to the left end with probability $\alpha\, dt$ independently of the size and configuration of the existing cluster. Similarly, a B-particle is added with probability $\beta\, dt$ at the right end. If there is an A-particle at the right end of the cluster it leaves the cluster with probability $\gamma_A\, dt$ in a time interval of length dt, and if there is a B-particle at the left end of the cluster it leaves the cluster with probability $\gamma_B\, dt$. This approach to the problem seems to be principally different from any earlier approach and that implies that the model does not relate directly to any previous work. A discussion of how the present work can be related to the existing literature will therefore be postponed to the end of the article (Sections 6 and 7). A general review of exact results for related models can be found in Schütz.[(15)]

We shall need some notation in order to define the mathematics of the model. We shall use n for the size of the cluster ($n = 0, 1, 2,...$). The particles in the cluster are numbered from left to right, $1, 2,..., n$. The type of the ith particle is described by an indicator variable, τ_i, which takes on the values 0 and 1—the value 0 means that the ith particle is a B-particle, while the value 1 means that it is an A-particle. The probability of finding the cluster with n particles given by $\tau_1, \tau_2,..., \tau_n$ to time t is given by

$$P_n(\tau_1, \tau_2,..., \tau_n; t).$$

The time evolution of the total set of probability functions is given by (compare Eq. (22) in Derrida *et al.*[(3)])

$$\begin{aligned}
\frac{d}{dt} P_n(\tau_1, \tau_2,..., \tau_n; t) = & -\sum_{i=1}^{n-1} \tau_i(1-\tau_{i+1})\, P_n(\tau_1,..., \tau_n; t) \\
& + \sum_{i=1}^{n-1} (1-\tau_i)\, \tau_{i+1} P_n(\tau_1,..., \tau_{i-1}, \tau_{i+1}, \tau_i, \tau_{i+2},..., \tau_n; t) \\
& -(\alpha+\beta)\, P_n(\tau_1,..., \tau_n) + \alpha\tau_1 P_{n-1}(\tau_2,..., \tau_n) \\
& + \beta(1-\tau_n)\, P_{n-1}(\tau_1,..., \tau_{n-1}) - \gamma_B(1-\tau_1)\, P_n(\tau_1,..., \tau_n) \\
& - \gamma_A \tau_n P_n(\tau_1,..., \tau_n) + \gamma_B P_{n+1}(0, \tau_1,..., \tau_n) \\
& + \gamma_A P_{n+1}(\tau_1,..., \tau_n, 1).
\end{aligned} \tag{1}$$

One might worry about the consequences of having the master equation working on an infinite set of states. A good source of information is the book by Feller.[6] The essential conclusion is that the only major difference from the finite case is that the existence of a stationary solution is no longer guaranteed. When we in the following try an ansatz for a stationary solution and the ansatz is not working in certain situations then there might be two reasons: Either the ansatz is wrong or the problem does not have a stationary solution.

3. THE STATIONARY SOLUTION

We shall start by looking for a stationary solution to (1), i.e., a set of functions $P_n(\tau_1,...,\tau_n)$ $(n=0, 1, 2,...)$ which when substituted for $P_n(\tau_1,..., \tau_n; t)$ in the right hand side of Eq. (1) give 0 for all values of n and all possible values of $\tau_1,...,\tau_n$. We shall make the same type of ansatz as Derrida *et al.*[3] One can hope that this will work because the size of the system does not occur in the ansatz (except through the number of matrices in the product).

Theorem 1. An unnormalized, stationary solution to Eq. (1) is given by

$$P_n(\tau_1,..., \tau_n) = \langle w| \prod_{i=1}^{n} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, |v\rangle, \tag{2}$$

where D and E are matrices which satisfy the following algebraic relation

$$DE = D + E, \tag{3}$$

while $\langle w|$ and $|v\rangle$ are vectors which multiply the matrices from left and right respectively. $\langle w|$ is a left eigenvector for E:

$$\langle w|\, E = (1/\gamma_B)\langle w|, \tag{4}$$

and $|v\rangle$ is a right eigenvector for D:

$$D\, |v\rangle = (1/\gamma_A)\, |v\rangle. \tag{5}$$

The algebra (Eqs. (3–5)) is the same as in Derrida *et al.*[3] and the proof which is given in Appendix A can therefore be modelled after their proof. The problem of finding matrices which satisfies Eq. (3) is easily solved. We shall use the 3rd set suggested by Derrida *et al.*[3] and take

$$D=\begin{Bmatrix} 1/\gamma_A & \sqrt{\eta} & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & & & & \ddots \end{Bmatrix} \qquad E=\begin{Bmatrix} 1/\gamma_B & 0 & 0 & 0 & \cdots \\ \sqrt{\eta} & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & \ddots & \ddots \end{Bmatrix}. \tag{6}$$

The (1,1)-elements are chosen to give the correct eigenvalues (Eqs. (4) and (5)). Also, the vectors $\langle w|$ and $|v\rangle$ are the same as in the third choice by Derrida *et al.*[3]

$$\langle w| = (1, 0, 0,\ldots) \qquad |v\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{7}$$

The undetermined element, η, is given by

$$\eta = (\gamma_A + \gamma_B - 1)/(\gamma_A \gamma_B). \tag{8}$$

The motivation for this choice for D and E is to ensure that the computation of $P_n(\tau_1,\ldots, \tau_n)$ only involves finite sums, which will ease the mathematics of the following computations.

Equations (2) and (6)–(8) determine a set of unnormalized probability functions (actually, they are normalized to make $P_0 = 1$). These functions are well defined for all values of α, β, γ_A, and γ_B. In order to make sense as probability functions they have to be normalized. To do that we have to calculate the normalization constant, N,

$$N = \sum_{n=0}^{\infty} \sum_{\tau_1, \tau_2,\ldots, \tau_n} P_n(\tau_1, \tau_2,\ldots, \tau_n). \tag{9}$$

It is only if the sum is finite that we have found a proper stationary solution. If we use Eq. (2) and define

$$C = \alpha D + \beta E, \tag{10}$$

then we can write Eq. (9) as

$$N = \sum_{n=0}^{\infty} \langle w|\, C^n \,|v\rangle. \tag{11}$$

Since C is bounded then we can always for sufficiently small x define

$$N(x) = \sum_{n=0}^{\infty} x^n \langle w|\, C^n \,|v\rangle = \langle w|\, (I - xC)^{-1} \,|v\rangle. \tag{12}$$

The operator $(I - xC)^{-1}$ is closely related to the resolvent of C (Kato[8]). If the spectral radius of C is less than 1 then the sum in Eq. (12) converges for $x = 1$ and $N = N(1)$ is finite and the probability functions can be normalized. If the spectral radius of C is larger than 1 then the sum in Eq. (12) diverges for $x = 1$ and we do not have a stationary solution. $N(x)$ can be computed explicitly; in Appendix B we show that

$$N(x)^{-1} = 1 - \left(\frac{\alpha}{\gamma_A} + \frac{\beta}{\gamma_B}\right) x + \frac{1}{2}\eta \left(-1 + (\alpha+\beta)\, x + \sqrt{1 - 2(\alpha+\beta)\, x + (\alpha-\beta)^2\, x^2}\right) \tag{13}$$

provided the square root is well defined and $N(x)^{-1}$ is not zero. The zeros of $N(x)^{-1}$ are most easily obtained by noticing that $N(x)^{-1}$ factorizes:

$$N(x)^{-1} = [1 - (\alpha x - \psi(x)/2)/\gamma_A][1 - (\beta x - \psi(x)/2)/\gamma_B], \tag{14}$$

where

$$\psi(x) = -1 + (\alpha+\beta)\, x + \sqrt{1 - 2(\alpha+\beta)\, x + (\alpha-\beta)^2\, x^2}. \tag{15}$$

The square root will be well defined unless x is real and

$$1 - 2(\alpha+\beta)\, x + (\alpha-\beta)^2\, x^2 < 0. \tag{16}$$

The cut defined in Eq. (16) is related to the continuous part of the spectrum of C, which comes from the infinite part of C (from the second row and downwards). The zeros of $N(x)^{-1}$ correspond to isolated eigenvalues of C. Depending on the values of the parameters, $N(x)^{-1}$ might have none, one or two zeros. The sum in (11) will converge if the cut (Eq. (16)) and the zeros of $N(x)^{-1}$ lie outside the closed unit disk in the complex x-plane.

Theorem 2. If the following conditions are met then the solution given in Theorem 1 will constitute a proper, normalizable stationary solution. The stationary solution is unique and is reached independent of the initial state.

We always have *Condition 1*:

$$\sqrt{\alpha} + \sqrt{\beta} \leqslant 1. \tag{17}$$

If $\gamma_A \geqslant 1$ and $\gamma_B \geqslant 1$ then there are no additional conditions. If $\gamma_A < 1$ then (*Condition 2A*) for $\gamma_A^2 \leqslant \alpha \leqslant \gamma_A$ one should have

$$\beta < (1-\gamma_A)(1-\alpha/\gamma_A). \tag{18}$$

If $\gamma_B < 1$ then (*Condition 2B*) for $\gamma_B^2 \leqslant \beta \leqslant \gamma_B$ one should have

$$\alpha < (1-\gamma_B)(1-\beta/\gamma_B). \tag{19}$$

The uniqueness and the fact that the stationary solution is always reached in the infinite time limit follows from the fact that the Markov process defined by Eq. (1) is irreducible (any state can be reached from any other state), Feller.[6] The proof of the existence is given in Appendix C. Condition 1 relates to the cut (Eq. (16)). Physically, the condition implies that the particles must not arrive so frequently that they are not able to pass each other sufficiently fast. Condition 2 comes from the zeros of $N(x)^{-1}$. Physically, it says that if γ_A is too small then the A-particles are not able to leave the cluster sufficiently fast and similarly with B-particles if γ_B is too small. It is worth noticing that conditions on γ_A and on γ_B are independent of each other because of the factorization (Eq. (14)). Also one should notice that for Condition 1 the boundary is included in the allowed region, while it is excluded in Condition 2. If $\gamma_A + \gamma_B \leqslant 1$ then the allowed region is determined by Condition 2 alone, otherwise Condition 1 will also play a role.

If α and β are small enough, i.e., if neither Condition 1 nor Condition 2 are violated, then the solution given by Theorem 1 is valid and the stable state can be compared to a gas state. The normalisation factor, $\tilde{N}$, is equal to $N(1)^{-1}$:

$$\tilde{N} = 1 - \left(\frac{\alpha}{\gamma_A} + \frac{\beta}{\gamma_B}\right) + \frac{1}{2}\eta\left(-1+\alpha+\beta+\sqrt{1-2\alpha-2\beta+(\alpha-\beta)^2}\right). \tag{20}$$

The size distribution of the cluster can be obtained by expanding $N(x)$ in powers of x. For example the chance of finding an empty cluster is

$$p_0 = \tilde{N}, \tag{21}$$

and the chance of finding a cluster with just one particle is

$$p_1 = \tilde{N}\left[\frac{\alpha}{\gamma_A} + \frac{\beta}{\gamma_B}\right]. \tag{22}$$

The average cluster size, $\langle n\rangle$, is found by differentiating $N(x)$ at $x=1$ and multiplying with $\tilde{N}$,

$$\langle n\rangle=\tilde{N}^{-1}\left(\frac{\alpha}{\gamma_A}+\frac{\beta}{\gamma_B}+\frac{1}{2}\eta\left(-\alpha-\beta+\frac{\alpha+\beta-(\alpha-\beta)^2}{\sqrt{1-2\alpha-2\beta+(\alpha-\beta)^2}}\right)\right). \tag{23}$$

The average number of A-particles in the cluster, $\langle n_A\rangle$, can be found by observing that α only enters C as a multiplicative factor to D. This means that $\langle n_A\rangle=\alpha\, \mathrm{d}N(1)/\mathrm{d}\alpha$ or

$$\langle n_A\rangle=\tilde{N}^{-1}\left(\frac{\alpha}{\gamma_A}+\frac{1}{2}\eta\left(-\alpha+\frac{\alpha-\alpha(\alpha-\beta)}{\sqrt{1-2\alpha-2\beta+(\alpha-\beta)^2}}\right)\right). \tag{24}$$

Similarly one finds for the average number of B-particles in the cluster, $\langle n_B\rangle$

$$\langle n_B\rangle=\tilde{N}^{-1}\left(\frac{\beta}{\gamma_B}+\frac{1}{2}\eta\left(-\beta+\frac{\beta+\beta(\alpha-\beta)}{\sqrt{1-2\alpha-2\beta+(\alpha-\beta)^2}}\right)\right). \tag{25}$$

If the boundary

$$\sqrt{\alpha}+\sqrt{\beta}=1, \tag{26}$$

or some part of it, is part of the allowed region, then the normalization factor, $\tilde{N}$, is larger than zero on the allowed part of Eq. (26) and the solution given by Eq. (13) remains valid. The probability of finding an empty cluster is still larger than zero. However, since the square root becomes zero, then the average cluster size and the average number of A-particles and the average number of B-particles all become infinite. The relative density, $\langle n_A\rangle/\langle n_B\rangle$, can still be defined,

$$\langle n_a\rangle/\langle n_B\rangle=\sqrt{\alpha/\beta}. \tag{27}$$

4. THE INFINITE CLUSTER

In this section we shall study the case where we are on the boundary of the "gas"-state, i.e., where Eq. (26) is satisfied and in order to not mess up things unnecessarily we shall assume that $\gamma_A\geqslant 1$ and $\gamma_B\geqslant 1$. In particular we shall look at the properties of the infinite cluster in the equilibrium state given by Theorem 1. This cluster can be considered as a dense state which is in equilibrium with a dilute state that supplies the incoming particles and absorbs the outgoing particles. We shall study the infinite cluster by looking at the cluster of size M in the limit $M\to\infty$. To be precise, from the probability distribution given by Eq. (2) we extract the conditional probabilities given that $n = M$ and compute the leading terms in the asymptotic expansions of the desired expectation values, valid in the limit $M \to \infty$. All the computations are given in Appendix D. If one takes $\gamma_A = \gamma_B = 1$, the expression for $N(x)^{-1}$ and all the computations of Appendix D are simplified considerably as can be seen in Derrida *et al.*[4]

The first step will be to calculate the normalization constant

$$N_M = \langle w| \, C^M \, |v\rangle. \tag{28}$$

We find in the limit of large M the following asymptotic value

$$N_M \simeq \frac{\eta \cdot (\alpha\beta)^{1/4}}{2\pi^{1/2}\tilde{N}^2} M^{-3/2}, \tag{29}$$

and

$$N_{M-1}/N_M \simeq 1 + (3/2)\, M^{-1}. \tag{30}$$

The next step will be to calculate

$$N_A(m, n) = \alpha\langle w| \, C^m D C^n \, |v\rangle, \qquad N_B(m, n) = \beta\langle w| \, C^m E C^n \, |v\rangle. \tag{31}$$

In the limit of large m and n we have the following asymptotic values

$$N_A(m, n) \simeq N_{m+n} \sqrt{\alpha}, \qquad N_B(m, n) \simeq N_{m+n} \sqrt{\beta}. \tag{32}$$

From this we conclude that in the bulk part of the infinite cluster the density of the A-particles is $\sqrt{\alpha}$ while the density of the B-particles is $\sqrt{\beta}$. By the bulk we understand any point specified such that the distance to both ends goes to infinity.

In the bulk we also have absence of correlations, i.e., that the probability of having a specified configuration on a finite sequence of points is equal to $\alpha^{na/2} \cdot \beta^{nb/2}$ where na and nb are the numbers of A- and B-particles in the specified configuration. This statement is also proved in Appendix D. In many respects this is the most important result of this section. Absence of correlation automatically gives that ratio of the rates is equal to the square of the ratio of the densities (Eq. (27)). It also implies that mean field calculations are correct.

It might seem like we are getting the wrong transport properties. The probability of finding an AB-pair is $\sqrt{\alpha\beta} = \sqrt{\alpha} - \alpha$, which implies that A-particles are transported over a fixed point on the line at that rate. But

A-particles are passed through the cluster at the rate α. Only if $\alpha = 1/4$ do the two values agree. The discrepancy comes from the fact if $\alpha \neq 1/4$ then the cluster moves as whole with the rate $\beta - \alpha = 1 - 2\sqrt{\alpha}$ towards the right. Since the density of A-particles is $\sqrt{\alpha}$ this subtracts $\sqrt{\alpha} - 2\alpha$ from the stream of A-particles through the cluster towards the right and gives the correct value.

One has the following exact value for the probability of having a B-particle at the left end of the cluster of size M, $(\beta/\gamma_B)\, N_{M-1}/N_M$. This implies that the B-particles leave the cluster at the rate of $\beta \cdot N_{M-1}/N_M \simeq \beta \cdot (1 + 3/(2M))$ per unit of time. Similarly the A-particles leave at the other end with the rate $\alpha \cdot N_{M-1}/N_M \simeq \alpha \cdot (1 + 3/(2M))$. The occurrence of a B-particle at the left end is uncorrelated with the occurrence of an A-particle at the right end.

5. THE UNBOUNDED GROWTH

If Eq. (32) is not satisfied, then one should expect the cluster to grow steadily and one can not expect to have a stable distribution of clusters in the infinite time limit. One would expect that the number of B-particles which leave the cluster per unit of time is given by the maximum allowed for the given value of A-particles which enters the cluster, i.e., the number of B-particles which leave the cluster is $(1 - \alpha^{1/2})^2$ per unit of time. Similarly, the number of A-particles which leave the cluster is $(1 - \beta^{1/2})^2$ per unit of time. The size of the cluster should thus increase with $\alpha + \beta - (1 - \alpha^{1/2})^2 - (1 - \beta^{1/2})^2$ per unit of time. And the relative density becomes

$$\frac{\langle n_a \rangle}{\langle n_B \rangle} = \frac{\alpha - (1 - \sqrt{\beta})^2}{\beta - (1 - \sqrt{\alpha})^2} = \frac{\sqrt{\alpha} + 1 - \sqrt{\beta}}{\sqrt{\beta} + 1 - \sqrt{\alpha}}. \tag{33}$$

All this are confirmed by some preliminary Monte Carlo calculations. A rigorous mathematical proof is outside the scope of the present article.

6. PARTICLES ON A RING

Most of the literature which might be related to present work concerns particles moving on a ring. One has a ring with a fixed number of sites, L, each site is either empty or occupied by one particle. One has two kinds of particles, A and B, which gives three possible states for each site, A, B, or E. The state of two neighbouring sites can be interchanged with rates which differ from model to model.

The choice by Evans *et al.*[5] stands out. They choose the rates for AB → BA, BE → EB, and EA → AE all to be 1 and the rates for BA → AB, EB → BE, and AE → EA to be q and prove that for $q \neq 1$ and equal numbers of A, B, and E one gets a segregation into three "phases," each dominated by one the species. A similar result has been obtained by Arndt *et al.*[1] and Arndt and Rittenberg[2] who made the choice AE → EA and EB → BE with rates λ, EA → AE and BE → EB with rates 0, BA → AB with rate 1 and AB → BA with rate q and by both Monte Carlo and mean field calculations found the same type of phase segregation for $q < 1$. The restrictions on the rates in this class of models are not consistent with the assumption that the particles are driven by an external field and it appears that the phase segregation seen here can not be related to observations on models which are consistent with the assumption of an external field as the driving force.

Arndt *et al.*[1] and Arndt and Rittenberg[2] also considered their model with $q > 1$ and found that if λ and the density of particles were large enough then the system segregated into two phases, one dense phase with no vacancies and one homogeneous phase with all the vacancies and some of the particles. Again they used both Monte Carlo and mean field calculations. Their results have been disputed by Rajewsky *et al.*[13] (see also Sasamoto and Zagier[14]) who performed a rigorous study based on the grand canonical ensemble. If one believes in the results by Arndt *et al.* then the dense phase they find can be compared to the infinite cluster in the present work and their phase transition between a one phase system at low density and a two phase system at high density is essentially the same transition found here when Eq. (26) is satisfied. A more detailed comparison is excluded since in the present work the backwards movement BA → AB is excluded.

Derrida *et al.*[4] gave a complete solution based on matrix algebra for the case where AE → EA, EB → BE, and AB → BA have rates one while the remaining rates are zero. They were interested in shock effects and focused on the situation with one or a few vacancies and interpreted the model in a different language, calling the vacancies for tracer particles and considering the B-particles as the vacancies. Their results on shock effects have no bearing on the present study, but as we shall see in the next section their mathematics has an interesting similarity to the present work. Mallick[10] studied the same problem with different rates for the three allowed interchanges.

Korniss *et al.*[9] and Mettetal *et al.*[11] made Monte Carlo calculations for a model with two parallel rings, where neighbouring sites on the two rings are allowed to interchange states. Along the rings the allowed interchanges are AE → EA and EB → BE with rates γ and AB → BA with rate 1. If one only had one ring then it follows from Rajewsky *et al.*[13] that one always will have one homogeneous phase. However, for the system with two rings and the right values of γ and the particle density Korniss *et al.*[9] found a two phase system similar to the findings of Arndt *et al.*[1] Again the dense phase they find can be compared to the infinite cluster in the present work. They only did computations for one density and one value of γ. But in that case they were able to reproduce the density in low density phase by a theoretical computation which assumed absence of correlation in the dense phase.

One might wonder how the "infinite cluster" which certainly has large size fluctuations according to the model can be used to describe the dense phase on a ring where size fluctuations are very small. Initially, one should notice that the fluctuations might well be larger measured in the lattice units, but if one scale with the size of the system they ought to be small. The mechanism behind this is the same as for the ordinary gas-liquid equilibrium. The number of particles which leave the infinite cluster per unit of time are independent of its size while the number which enter depends on the concentration of particles in the thin phase. If the infinite cluster is too small there are too many particles in the thin phase and more particles enter the infinite cluster than leave and vice versa.

7. DISCUSSION

It might appear more natural to compare with the computations on open systems (which are actually clusters of a fixed size, but with variable densities) like the computations by Derrida *et al.*[3] In fact, fixing the size, changes the restrictions on the dynamics so drastically that the comparison is difficult. Derrida *et al.*[3] find three different states, the maximum current state, an A-rich state and a B-rich state. They mostly resemble the infinite cluster in the present work with $\alpha = \beta = 1/4$ being the maximum current state, while the part of Eq. (26) where $\alpha > \beta$ corresponds to the A-rich state and the part with $\alpha < \beta$ corresponds to the B-rich state. Clearly the geometry of the phase space is very different in the two cases.

As mentioned earlier there exists an interesting connection to the grand canonical ensemble used in some studies of dynamical systems. If one takes $\gamma_A = \gamma_B = 1$ then the generating function for the probability distribution, $N(x)$, is identical to the grand canonical ensemble in Derrida *et al.*[4] with α and β playing the role of the fugacities. In an ideal gas the fugacity is equal to the pressure and the mechanical definition of the pressure says that it is proportional to the number of particle that hit the surface per unit time. So it looks like more than an amusing coincidence.

To sum it up: A dynamical one dimensional model for a system with variable cluster size has been introduced. It has been proven that it has a behaviour which in some aspects looks like a liquid-gas transition. A complete solution has been given for the gaseous state up to and including the point of phase transition. It has been proven that the condensed state (the infinite cluster) at the transition point is homogeneous and has no correlation, not even short range correlation. It has been argued that the infinite cluster is a model for the condensed state in certain models on a ring. But perhaps one should concentrate on the present model with variable cluster size, which after all looks much more like something which could be realized experimentally.

Nothing has been proven about the behaviour beyond the transition point. That also goes for the transitions at low values of γ_A and/or γ_B.

APPENDIX A: PROOF OF THEOREM 1

Equation (2) can alternatively be written as

$$P_n(\tau_1,..., \tau_n) = \langle w| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i)\, E)\, |v\rangle\, \alpha^{n_A} \beta^{n_B}, \tag{A1}$$

where

$$n_A = \sum_{i=1}^{n} \tau_i \qquad \text{and} \qquad n_B = \sum_{i=1}^{n} (1-\tau_i). \tag{A2}$$

n_A and n_B are of course not changed by the internal movements in the cluster. These movements are represented by the two sums in Eq. (1) and it is shown by Derrida *et al.*[3] that the assumption in Eq. (3) reduces the sums to two single terms, one for each end. This is not changed by the factors α^{nA} and β^{nB} and we end up we the following condition for stationarity

$$\begin{aligned}
0 &= (\alpha\tau_1 - x(\tau_1))\, \langle w| \prod_{i=2}^{n} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, |v\rangle \\
&\quad + (\beta(1-\tau_n) + x(\tau_n))\, \langle w| \prod_{i=1}^{n-1} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, |v\rangle \\
&\quad - (\alpha + \beta + \gamma_B(1-\tau_1) + \gamma_A \tau_n)\, \langle w| \prod_{i=1}^{n} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, |v\rangle \\
&\quad + \gamma_B \beta \langle w|\, E \prod_{i=1}^{n} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, |v\rangle \\
&\quad + \gamma_A \alpha \langle w| \prod_{i=1}^{n} (\tau_i \alpha D + (1-\tau_i)\, \beta E)\, D\, |v\rangle
\end{aligned} \tag{A3}$$

where

$$x(\tau) = \tau\alpha + (\tau - 1)\,\beta. \tag{A4}$$

As a first step towards finding a solution to Eq. (A4) we shall split it into two parts. One concerned with the left end

$$\begin{aligned} 0 = (\alpha\tau_1 - x(\tau_1))\,\langle w| \prod_{i=2}^{n} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,|v\rangle \\ -(\delta + \gamma_B(1-\tau_1))\,\langle w| \prod_{i=1}^{n} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,|v\rangle \\ +\gamma_B\beta\langle w|\, E \prod_{i=1}^{n} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,|v\rangle \end{aligned} \tag{A5}$$

and one concerned with the right end

$$\begin{aligned} 0 = (\beta(1-\tau_n) + x(\tau_n))\,\langle w| \prod_{i=1}^{n-1} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,|v\rangle \\ -(\alpha + \beta - \delta + \gamma_A\tau_n)\,\langle w| \prod_{i=1}^{n} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,|v\rangle \\ +\gamma_A\alpha\langle w| \prod_{i=1}^{n} (\tau_i\alpha D + (1-\tau_i)\,\beta E)\,D\,|v\rangle \end{aligned} \tag{A6}$$

where δ is a yet undetermined parameter. For fixed values of $\tau_2,\ldots,\tau_n$ Eq. (A5) splits into two conditions, one for $\tau_1 = 0$

$$0 = \langle w|\,(\beta - (\delta + \gamma_B)\,\beta E + \gamma_B\beta^2 E^2), \tag{A7}$$

and one for $\tau_1 = 1$

$$0 = \langle w|\,(-\delta\alpha D + \gamma_B\alpha\beta ED). \tag{A8}$$

Similarly, for fixed values of $\tau_1,\ldots,\tau_{n-1}$ Eq. (A6) splits into two conditions, one for $\tau_n = 0$

$$0 = (-(\alpha + \beta - \delta)\,\beta E + \gamma_A\alpha\beta ED)\,|v\rangle, \tag{A9}$$

and one for $\tau_n = 1$

$$0 = (\alpha - (\alpha + \beta - \delta + \gamma_A)\,\alpha D + \gamma_A\alpha^2 D^2)\,|v\rangle. \tag{A10}$$

Using the eigenvalue equations (4) and (5) it is easily checked that with the choice $\delta = \beta$ all 4 equations are satisfied.

APPENDIX B: PROOF OF EQ. (13)

We want to calculate the (1,1)-element of $(I-xC)^{-1}$. We shall do this by a method which was introduced many years ago for the calculation of relaxation spectra (Heilmann[7]). We start by calculating the (1,1)-element of $(I_M-xC_M)^{-1}$, the matrix where we have only taking the first M columns and rows, and then afterwards taking the limit $M\to\infty$. The elements of an inverse matrix are given by the cofactor divided by the determinant. Determinants of tridiagonal matrices are easily calculated by recursion. We shall use $P_M(x)$ for the cofactor of the (1,1)-element and $Q_M(x)$ for the determinant of I_M-xC_M. For $M>2$ both P_M and Q_M will satisfy the same recurrence relation (here written out for P_M):

$$P_M(x)=(1-(\alpha+\beta)\,x)\,P_{M-1}(x)-\alpha\beta x^2P_{M-2}(x). \tag{B1}$$

The initial conditions are

$$P_1(x)=1, \qquad P_2(x)=1-(\alpha+\beta)\,x, \tag{B2}$$

and

$$Q_1(x)=1-\xi x, \qquad Q_2(x)=(1-(\alpha+\beta)\,x)\,Q_1(x)-\alpha\beta\eta x^2, \tag{B3}$$

where

$$\xi=\frac{\alpha}{\gamma_A}+\frac{\beta}{\gamma_B}. \tag{B4}$$

The computation of the cofactor is standard. If we write

$$\overline{P_M}(y)=(x\sqrt{\alpha\beta})^{1-M}\,P_M(x), \tag{B5}$$

and

$$y=(1-(\alpha+\beta)\,x)/(2x\sqrt{\alpha\beta}), \tag{B6}$$

then the recurrence relation (B1) transforms into the recurrence relation for Tchebichef polynomials. The initial conditions (B2) allow the identification

$$\overline{P_M}(y)=U_{M-1}(y), \tag{B7}$$

where $U_M(y)$ are the Tchebichef polynomials of the second kind. The determinants are then found by noticing that if we define $P_0 = 0$, which is consistent with the Tchebichef polynomials, then

$$Q_M(x) = (1-\xi x)\, P_M(x) - \alpha\beta\eta x^2 P_{M-1}(x). \tag{B8}$$

Since x is small and y therefor is large, then it is more practical to identify the Tchebichef polynomials with the hyperbolic functions rather than the circular functions. We have

$$P_M(x) = (x\sqrt{\alpha\beta})^{M-1} \sinh(Mz)/\sinh(z), \tag{B9}$$

where

$$\cosh(z) = (1-(\alpha+\beta)\, x)/(2x\sqrt{\alpha\beta}), \tag{B10}$$

If y is larger than 1, then we can solve (B10) for e^z and choose the solution where the real part of z is larger than zero. We write the solution as

$$e^{-z} = \frac{1-(\alpha+\beta)\, x - \sqrt{1-2(\alpha+\beta)\, x + (\alpha-\beta)^2\, x^2}}{2x\sqrt{\alpha\beta}}. \tag{B11}$$

According to our plan we should calculate $N(x)$ as

$$N(x) = \lim_{M\to\infty} P_M(x)/Q_M(x). \tag{B12}$$

It is easily checked that Eq. (13) obtains.

APPENDIX C: PROOF OF THEOREM 2

The condition (16) is resolved by finding the numerically smallest zero of

$$1-2(\alpha+\beta)\, x + (\alpha-\beta)^2\, x^2 = 0.$$

This is easily seen to be

$$x = \frac{\alpha+\beta-\sqrt{4\alpha\beta}}{(\alpha-\beta)^2},$$

provided $\alpha \neq \beta$. If $\alpha = \beta$ then it is $x = 1/(4\alpha)$. In any case one easily confirms that the requirement that this zero should not be smaller than one leads to Condition 1.

To prove Condition 2A we start by finding the possible solutions to

$$\gamma_A - \alpha x + \psi(x)/2 = 0.$$

It easily seen that the only possible solution is

$$x = \gamma_A(1-\gamma_A)/[\alpha(1-\gamma_A)+\gamma_A\beta],$$

and this will only be a solution if

$$\alpha(1-\gamma_A)^2 \geqslant \beta\gamma_A^2.$$

The requirement that the zero should be larger than one gives Condition 2A. Condition 2B is proven in the same way.

APPENDIX D: CALCULATIONS FOR SECTION 4

We start by getting the first term of the asymptotic expansion for N_M. The residue theorem gives

$$N_M = \frac{1}{2\pi i} \oint dx\, N(x)/x^{M+1}, \tag{D1}$$

where the integral is around a small circle in the complex x-plane with centre at $x=0$. The singularities of $N(x)$ will be the cut (Eq. (16)) and the possible zeros of $N(x)^{-1}$. Equation (26) ensures that the cut starts at $x=1$ and continues out along the real x-axis. The conditions on γ_A and γ_B ensure that possible zeros will occur for $|x|>1$, so that the dominant contribution for large M comes from the start of the cut at $x=1$. The contribution from the cut is

$$N_M \simeq \frac{1}{\pi}\int_1^b \exp(-M\ln(x))\, \frac{(x-1)^{1/2}\, g(x)^{1/2}}{f(x)^2+(x-1)\, g(x)}\, \frac{1}{x}\, dx, \tag{D2}$$

where

$$f(x) = 1-[(\alpha/\gamma_A)+(\beta/\gamma_B)]\, x + \tfrac{1}{2}\eta[-1+(\alpha+\beta)\, x], \tag{D3}$$

and

$$g(x) = (\eta/2)^2\, [4\sqrt{\alpha\beta}-(x-1)(\alpha-\beta)^2]. \tag{D4}$$

Using Laplace's method (Olver[12]) one finds Eq. (29) ($f(1)=\tilde{N}$). One can if necessary find the following terms in the asymptotic expansion, but we do not need them; the next term is of order $M^{-5/2}$ and that is sufficient to prove Eq. (30).

Lemma D1.

$$D^nC=\alpha D^{n+1}+\beta\sum_{j=0}^{n-1}D^{n-j}+\beta E. \tag{D5}$$

Proof. The proof is by induction on n. For $n=0$ it is obviously true: the induction follows easily from Eq. (3). ∎

Lemma D2. We can write

$$C^n=\sum_{p=1}^{n}\sum_{q=0}^{p}x_{p,q}(n)\,E^qD^{p-q}, \tag{D6}$$

where ($q=0,\dots,n$)

$$x_{n,q}(n)=\alpha^{n-q}\beta^q, \tag{D7}$$

and

$$x_{n-r,q}(n)=\sum_{j=0}^{r}\alpha^{n-q-j}\beta^{q+j}\left[\binom{n-q-1}{j}\binom{q+r-1}{r-j}-\binom{n-q-1}{j-1}\binom{q+r-1}{r-j-1}\right], \tag{D8}$$

$r=1,\dots,n-1$ and $q=0,\dots,n-r$. If $q=n-r$ then the upper limit on the summation should be $n-q-1$.

Proof. The proof is again by induction on n. For $n=1$ the statement is clearly correct. For the induction we start by multiplying Eq. (D6) from the right with C, using Lemma D1. After rearranging the sums one finds the recursion relations for $x_{p,q}(n)$. The fulfilment will be a necessary and sufficient condition for the induction to work.

$$x_{q,q}(n+1)=\beta\sum_{j=q-1}^{n}x_{j,q-1}(n),$$

for $q=2,\dots,n+1$; for $q=1$ one gets the same expression except that summation should start with $j=1$. For $p>q$ the relation becomes

$$x_{p,q}=\alpha x_{p-1,q}(n)+\beta\sum_{j=p}^{n}x_{j,q}(n),$$

except for $p=1$ where the first term should be left out. However the two exceptions do not matter because Eq. (D8) gives $x_{0,0}(n)=0$. That the recursion relations are consistent with Eqs. (D7) and (D8) is seen by substituting (D7) and (D8) on the right hand side, interchanging the summations and use

$$\sum_{j=0}^{n} \binom{m+j}{j} = \binom{n+m+1}{m+1}. \qquad \blacksquare$$

Lemma D3. If we define vectors $|a_q\rangle$ $(q=0, 1, 2,...)$ with components $a_{q,r}$ $(r=0, 1, 2,...)$:

$$|a_q\rangle = E^q\,|v\rangle, \tag{D9}$$

then

$$\begin{aligned} a_{q,0} &= (\beta/\gamma_B)^q \\ a_{q,r} &= \sqrt{\eta}\sum_{j=0}^{q-r}\binom{q-1-j}{r-1}(\beta/\gamma_B)^j, \qquad \text{for} \quad 1\leqslant r\leqslant q \\ a_{q,r} &= 0, \qquad \text{for} \quad r>q. \end{aligned} \tag{D10}$$

Proof. The first and the last line of Eq. (D10) are trivial consequences of the definitions of E and $|v\rangle$ (Eqs. (6) and (7)). The middle line follows by induction on q, taking $r=1$ first and then $r>1$. $\blacksquare$

We also define vectors $|b_n\rangle$ $(n=0, 1, 2,...)$ with components $b_{n,r}$ $(r=0, 1, 2,...)$:

$$|b_n\rangle = C^n\,|v\rangle. \tag{D11}$$

Lemma D4. We have

$$b_{n,0} = N_n, \tag{D12}$$

and (ξ is defined in Eq. (B3))

$$b_{n,1} = (N_{n+1} - \xi N_n)/(\alpha\sqrt{\eta}). \tag{D13}$$

Proof. Equation (D12) is a simple consequence of the definitions. For Eq. (D13) we use

$$\langle v|\,EC^n\,|w\rangle = N_n/\gamma_B, \qquad \langle v|\,DC^n\,|w\rangle = b_{n,0}/\gamma_A + b_{n,1}\sqrt{\eta},$$

which together with $C=\alpha D+\beta E$ gives the desired result. $\blacksquare$

Lemma D5. More generally we have for $1 \leqslant r \leqslant n$ (the integral is in the complex w-plane along a small circle around the origin)

$$b_{n,r} = \sqrt{\eta} \sum_{k=0}^{n-r} \binom{r-1+k}{r-1} \beta^{r+k} \frac{1}{2\pi i} \oint dw(1+w)^n (\beta+\alpha/w)^{n-r-k}$$
$$(\alpha - w^2\beta)\, w^{-1}[1+w-w/\gamma_A]^{-1} [\alpha+\beta w(1-1/\gamma_B)]^{-1}. \quad \text{(D14)}$$

For $r > n$ we have $b_{n,r} = 0$.

Proof. We start by using Eq. (D6), Lemma D3 and $D^j |w\rangle = \gamma_A^{-j}$ to get

$$b_{n,r} = \sqrt{\eta} \sum_{k=0}^{n-r} \binom{r-1+k}{r-1} \sum_{p=r+k}^{n} \sum_{q=r+k}^{p} x_{p,q}(n)\, \gamma_B^{r+k-q} \gamma_A^{q-p}. \quad \text{(D15)}$$

If one expands $(1+w)^{q+r-1} (\alpha+\beta w)^{n-q-1}$ in powers of w and compare with Eqs. (D7) and (D8) then one can see that one has the expression for $x_{p,q}(n)$ in terms of a contour integral

$$x_{p,q}(n) = \frac{\beta^q}{2\pi i} \oint dw(1+w)^{q+n-p-1} (\alpha+\beta w)^{n-q-1} (\alpha - \beta w^2)\, w^{p-n-1}. \quad \text{(D16)}$$

Using this in (D15) the summations on p and q can be done explicitly. We start with the summation on p. It can be extended to infinity since the contour integral gives zero if $p > n$ and the summation converges absolutely if $|w|$ is sufficiently small. Afterwards the same argument applies to the summation on q and the lemma obtains. ∎

We define the vectors $\langle \tilde{b}_n|$ $(n = 0, 1, 2,...)$ with components $\tilde{b}_{n,r}$ $(r = 0, 1, 2,...)$:

$$\langle b_n| = \langle w|\, C^n. \quad \text{(D17)}$$

Lemma D6. We have

$$\tilde{b}_{n,0} = N_n, \quad \text{(D18)}$$

$$\tilde{b}_{n,1} = (N_{n+1} - \xi N_n)/(\beta \sqrt{\eta}), \quad \text{(D19)}$$

$$\tilde{b}_{n,r} = \sqrt{\eta} \sum_{k=0}^{n-r} \binom{r-1+k}{r-1} \alpha^{r+k} \frac{1}{2\pi i} \oint dw(1+w)^n (\alpha+\beta/w)^{n-r-k}$$
$$(\beta - w^2\alpha)\, w^{-1}[1+w-w/\gamma_B]^{-1} [\beta+\alpha w(1-1/\gamma_A)]^{-1}. \quad \text{(D20)}$$

Proof. Since $\langle \tilde{b}_n |$ obtains from $| b_n \rangle$ by transposing followed by an interchange of α and β and γ_A and γ_B then lemma follows trivially from the preceding lemmas. ▮

It follows from the definitions that we have

$$\langle \tilde{b}_m \mid b_n \rangle = N_{m+n}, \tag{D21}$$

$$\sum_{r=1}^{[n,m]} \tilde{b}_{m,r} b_{n,r} = N_{m+n} - N_m N_n. \tag{D22}$$

We shall also need to calculate

$$c_{m,n} = \alpha \sum_{r=1}^{[m,n-1]} \tilde{b}_{m,r} b_{n,r+1} \tag{D23}$$

and

$$\tilde{c}_{m,n} = \beta \sum_{r=1}^{[m-1,n]} \tilde{b}_{m,r+1} b_{n,r}. \tag{D24}$$

Lemma D7. We have

$$c_{m,n} = \frac{\eta}{(2\pi i)^2} \oint dy \oint dx\, y^{-m-1} x^{-n-1} N(y)\, N(x)$$
$$[\alpha\beta xy + \psi(x)\, \psi(y)/4 - (\alpha+\beta-1/x)\, y\psi(x)/2]/(y-x). \tag{D25}$$

Proof. We start with Eqs. (D14) and (D20) with the summation variable changed $j = k + r$

$$c_{m,n} = \eta \sum_{r=1}^{[m,n-1]} \sum_{j'=r}^{m} \sum_{j=r}^{n-1} \binom{j'-1}{r-1} \binom{j}{r} \alpha^{j'+1} \beta^{j+1} \frac{1}{(2\pi i)^2} \oint dv \oint dw$$
$$(1+v)^m (\alpha+\beta/v)^{m-j'} (\beta - v^2\alpha)$$
$$v^{-1}[1+v-v/\gamma_B]^{-1} [\beta + \alpha v(1-1/\gamma_A)]^{-1}$$
$$(1+w)^n (\beta+\alpha/w)^{n-j-1} (\alpha - w^2\beta)$$
$$w^{-1}[1+w-w/\gamma_A]^{-1} [\alpha + \beta w(1-1/\gamma_B)]^{-1}.$$

The summation on r can be done explicitly after reordering of the summations

$$\sum_{r=1}^{[j,j']} \binom{j'-1}{r-1}\binom{j}{r} = \binom{j+j'-1}{j-1}.$$

The binomial can be represented with a contour integral in the complex t-plane around $t=0$ with the restriction $|t|<1$

$$\binom{j+j'-1}{j-1} = \frac{1}{2\pi i}\oint dt (1-t)^{-j'-1}\, t^{-j}.$$

If the upper limit for the summation on j' is extended then the contour integral on v will ensure that the additional terms are all zero, and if $|v|$ is sufficiently small then the sum on j' will converge absolutely. More precisely we can take $|t|=\frac{1}{2}$ and $|v|=\min\{\beta/(4\alpha), 1\}$. We can consequently extend the upper limit to ∞. Similarly, we extend the summation on j to ∞ if we also take $|w|=\min\{\alpha/(4\beta), 1\}$. Now the summations on j and j' can be done explicitly

$$\begin{aligned} c_{m,n} = \frac{\eta\alpha^2\beta^2}{(2\pi i)^3}\oint dt \oint dv \oint dw (1-t)^{-1}\,(1-t-\alpha v/(\alpha v+\beta))^{-1}\,(t-\beta w/(\beta w+\alpha))^{-1} \\ (1+v)^m\,(\alpha+\beta/v)^{m-1}\,(\beta-v^2\alpha)\, v^{-1}[1+v-v/\gamma_B]^{-1}\,[\beta+\alpha v(1-1/\gamma_A)]^{-1} \\ (1+w)^n\,(\beta+\alpha/w)^{n-2}\,(\alpha-w^2\beta)\, w^{-1}[1+w-w/\gamma_A]^{-1}\,[\alpha+\beta w(1-1/\gamma_B)]^{-1}. \end{aligned}$$

The integration on t can easily be done. With the above restrictions there is just one pole, $t=\beta w/(\beta w+\alpha)$, inside $|t|=\frac{1}{2}$

$$\begin{aligned} c_{m,n} = \frac{\eta\beta}{(2\pi i)^2}\oint dv \oint dw (1-vw)^{-1}\,[(1+v)(\alpha+\beta/v)]^m\,(\beta-v^2\alpha) \\ [1+v-v/\gamma_B]^{-1}\,[\beta+\alpha v(1-1/\gamma_A)]^{-1}\,[(1+w)(\beta+\alpha/w)]^n \\ (\alpha-w^2\beta)\, w[1+w-w/\gamma_A]^{-1}\,[\alpha+\beta w(1-1/\gamma_B)]^{-1}. \end{aligned}$$

The next step is to change the integration variables to x and y

$$\begin{aligned} x &= w/[(1+w)(\beta w+\alpha)], \\ y &= v/[(1+v)(\alpha v+\beta)]. \\ w &= -\psi(x)/(2x\beta), \\ v &= -\psi(y)/(2y\alpha). \end{aligned}$$

If $|v|$ and $|w|$ are small enough there is no problems with the mapping being one-to-one and the closed contours around the origin in the v- and w-planes map onto closed contours around theorigin in the y- and x-planes respectively. We have

$$(\alpha^2-\beta w^2)\,dw=(w^2/x^2)\,dx,$$
$$[1+w(1-1/\gamma_A)][\alpha+\beta w(1-1/\gamma_B)]=(w/x)\,N(x)^{-1},$$

and similarly for the connection between v and y. Finally, we have

$$\frac{\beta v w^2}{1-vw}=\frac{1}{y-x}\,[\alpha\beta xy+\psi(x)\,\psi(y)/4-(\alpha+\beta-1/x)\,y\psi(x)/2]. \tag{D26}$$

From which the proof is easily concluded. ▮

Lemma D8. We have

$$\tilde{c}_{m,n}=\frac{\eta}{(2\pi i)^2}\oint dy\oint dx\; y^{-m-1}x^{-n-1}N(y)\,N(x)$$
$$[\alpha\beta xy+\psi(x)\,\psi(y)/4-(\alpha+\beta-1/y)\,x\psi(y)/2]/(x-y). \tag{D27}$$

Proof. By analogy to Lemma D7. ▮

Lemma D9. If $m\leqslant n$ and $m\to\infty$, $n\to\infty$ then

$$c_{m,n}-\tilde{c}_{m,n}=O((m+n)^{-3/2}\,m^{-1/2}). \tag{D28}$$

Proof. Since the left side of Eq. (D26) does not have any singularities for $|vw|<1$, then the right side can not have any singularities for $x=y$ and that goes for the integrands of Eqs. (D25) and (D27) too. Subtracting the integrals as they stand in Eqs. (D25) and (D27) and then adding the whole once more with x and y interchanged we get

$$2(c_{m,n}-\tilde{c}_{m,n})=\frac{\eta}{(2\pi i)^2}\oint dx\oint dy[x^{-n-1}y^{-m-1}-x^{-m-1}y^{-n-1}](y-x)^{-1}$$
$$[2\alpha\beta xy+\psi(x)\,\psi(y)/2-(\alpha+\beta-1/x)\,y\psi(x)/2$$
$$-(\alpha+\beta-1/y)\,x\psi(y)/2]\,N(x)\,N(y).$$

The first line of the integrand gives

$$[x^{-n-1}y^{-m-1}-x^{-m-1}y^{-n-1}](y-x)^{-1}=\sum_{j=0}^{n-m}x^{j-n-1}y^{-j-m-1}.$$

The last two lines give

$$\begin{aligned}&(2/\eta^2)+N(x)[2(\xi x-1)/\eta^2-(\alpha+\beta-1/y)\,x/\eta]\\&\quad+N(y)[2(\xi y-1)/\eta^2-(a+\beta-1/x)\,y/\eta]\\&\quad+[2\alpha\beta xy+2(1-\xi x)(1-\xi y)/\eta^2+(\alpha+\beta-1/x)\,y(1-\xi x)/\eta\\&\quad+(\alpha+\beta-1/y)\,x(1-\xi y))/\eta]\,N(x)\,N(y).\end{aligned}$$

The first line of the above gives 0 when integrated, while the last two lines to the lowest order in m and n give when we use Eqs. (D1) and (29)

$$c_{m,n}-\tilde{c}_{m,n}=A\sum_{j=0}^{n-m}(n-j)^{-3/2}\,(m+j)^{-3/2},$$

for some constant A which does not depend on m and n. The sum can be estimated by the integral

$$\int_0^{n-m}dj[(m+j)(n-j)]^{-3/2}=4(n+m)^{-2}\,(n-m)(nm)^{-1/2},$$

from which the lemma obtains. ▮

Proof of Eq. (32). Expanding D, taking the element of the first row separately and the diagonal and the superdiagonal of rest one gets

$$N_A(m,n)=(\alpha/\gamma_A)\,N_mN_n+N_m(N_{n+1}-\xi N_n)+\alpha(N_{n+m}-N_mN_m)+c_{m,n},$$

where we have used Lemma D4, Lemma D6, and Eqs. (D22) and (D23). The definition implies

$$N_A(m,n)+N_B(m,n)=N_{m+n+1}.$$

Lemma D9 together with the expansion above imply

$$N_A(m,n)-N_B(m,n)\simeq(\alpha-\beta)\,N_{m+n},$$

to the leading order in m and n. Solving the two equations for $N_A(m,n)$ and $N_B(m,n)$ to the same order, using Eq. (26), gives Eq. (32). ▮

Proof of the Absence of Correlation. We specify a given sequence of k particles by indicator variables, $\tau_1,\tau_2,\ldots,\tau_k$, just as in Section 2. We introduce unnormalized probabilities by

$$N(m,n;\tau_1,\tau_2,\ldots,\tau_k)=\langle w|\,C^m\prod_{j=1}^{k}(\tau_j\alpha D+(1-\tau_j)\,\beta E)\,C^n\,|v\rangle.$$

Equation (32) implies that to the leading order in m and n we have for $k=1$

$$N(m, n; 1) = \sqrt{\alpha}\, N_{m+n}, \qquad N(m, n; 0) = \sqrt{\beta}\, N_{m+n}.$$

For $k=2$ the algebra gives

$$\begin{aligned} N(m, n; 1, 0) &= \beta N(m, n; 1) + \alpha N(m, n; 0) \simeq (\beta\sqrt{\alpha} + \alpha\sqrt{\beta})\, N_{m+n} \\ &= \sqrt{\alpha\beta}\, N_{m+n}. \end{aligned}$$

We shall call this argument the algebraic argument. The definitions imply

$$N(m, n; 1, 0) + N(m, n; 1, 1) = N(m, n+1; 1) \simeq \sqrt{\alpha}\, N_{m+n}$$

or

$$N(m, n; 1, 1) \simeq \alpha N_{m+n}.$$

We shall call this argument the "definition" argument. The same argument applied to τ_1 gives

$$N(m, n; 0, 0) \simeq \beta N_{m+n}, \qquad N(m, n; 0, 1) \simeq \sqrt{\alpha\beta}\, N_{m+n}.$$

If we define for any k and any configuration

$$na = \sum_{j=1}^{k} \tau_j, \qquad nb = k - na,$$

then we have proved that for $k=1$ and 2 we have

$$N(m, n; \tau_1, \tau_2, \ldots, \tau_k) \simeq \alpha^{na/2} \beta^{nb/2} N_{m+n}. \tag{D29}$$

We shall proceed with induction to prove that it holds for all k, using the same two arguments again. We assume Eq. (D29) to be true for all configurations of length k. For a configuration of length $k+1$ which ends with $\tau_k = 1$ and $\tau_{k+1} = 0$ the algebraic argument can be used to reduce the problem to configurations of length k. We can then use the "definition" argument to handle configurations of length $k+1$ which end with $\tau_k = 1$ and $\tau_{k+1} = 1$. Similarly, for configurations of length $k+1$ which starts with $\tau_1 = 1$ and $\tau_2 = 0$ the algebraic argument can be used to reduce the problem to configurations of length k, and then the "definition" argument to handle configurations of length $k+1$ which start with $\tau_1 = 0$ and $\tau_2 = 0$. So we are left with the problem of handling configurations of length $k+1$ which have $\tau_2 = 1$ and $\tau_k = 0$. If $k = 2$ no such configurations exist. If $k > 2$ we can find a j $(2 \leqslant j < k)$ such that $\tau_j = 1$ and $\tau_{j+1} = 0$ and then apply the algebraic argument to that pair. This concludes the induction. ▮

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